4.7. Teori Transformasi Formalisme Bra-Ket

Formalisme ket — calon set ket asas melebihi satu
e.g. set ket eigen dua operator hermite $\hat{A}$ dan $\hat{B}$

$$\hat{A}\,|\,a_i\rangle \;=\; a_i\,|\,a_i\rangle \quad ; \quad \hat{B}\,|\,b_j\rangle \;=\; b_j\,|\,b_j\rangle \quad , \quad (i, j \;=\; 1, 2, \cdots, n) \quad .$$

Persoalan:

i) Apakah hubungan objek-objek ket-bra dan operator dalam kedua-dua set ket asas yang ada?
ii) Bagaimanakah teori kuantum yang akan dibina dijamin unik tanpa bergantung kepada pemilihan set ket asas yang ada?

Tahu: sebarangan ket dapat ditulis dlm sebutan hasiltambah linear set ket asas

$$|\,a_i\rangle \;=\; \sum_{j=1}^{n} |\,b_j\rangle\,\langle b_j\,|\,a_i\rangle \;=\; \sum_{j=1}^{n} |\,b_j\rangle\, U_{ji} \quad ,$$

Unsur matriks transformasi dari set $\{\,|\,a_i\rangle\,\}$ ke set $\{\,|\,b_j\rangle\,\}$:

$$U_{ji} \;:=\; \langle b_j\,|\,a_i\rangle$$

Cara sama: transformasi sebaliknya

$$|\,b_j\rangle \;=\; \sum_{k=1}^{n} |\,a_k\rangle\,\langle a_k\,|\,b_j\rangle \;=\; \sum_{k=1}^{n} |\,a_k\rangle\,\langle b_j\,|\,a_k\rangle^* \;=\; \sum_{k=1}^{n} |\,a_k\rangle\, U_{jk}^* \;=\; \sum_{k=1}^{n} |\,a_k\rangle\,(U_{kj}^*)^t \;=\; \sum_{k=1}^{n} |\,a_k\rangle\, U_{kj}^\dagger \quad .$$

$\Longrightarrow$ matriks $U^\dagger$ bagi transformasi dari set $\{\,|\,b_j\rangle\,\}$ ke set $\{\,|\,a_i\rangle\,\}$

Masukkan ungkapan $|b_j\rangle$ ke dalam ungkapan $|a_i\rangle$:

$$|a_i\rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} |a_k\rangle U^\dagger_{kj} U_{ji} \quad .$$

Menggunakan sifat tak bersandar linear set $\{ |a_i\rangle \}$:

$$\sum_{j=1}^{n} U^\dagger_{kj} U_{ji} = \delta_{ki} \quad \Longrightarrow \quad U^\dagger U = I \quad .$$

Matriks/operator transformasi = matriks/operator unitari

$$U^\dagger U = \hat{1} \quad \Rightarrow \quad U^\dagger = U^{-1} \quad .$$

Transformasi ket umum?

Lihat unsurnya

$$\langle b_i | \psi \rangle = \sum_{j=1}^{n} \langle b_i | a_j \rangle \langle a_j | \psi \rangle$$

$$= \sum_{j=1}^{n} U_{ij} \langle a_j | \psi \rangle \quad .$$

Bentuk matriks: $|\psi\rangle_{(b)} = U |\psi\rangle_{(a)}$.

Transformasi songsangan:

$$\langle a_j | \psi \rangle = \sum_{k=1}^{n} \langle a_j | b_k \rangle \langle b_k | \psi \rangle$$

$$= \sum_{k=1}^{n} U^\dagger_{jk} \langle b_k | \psi \rangle \quad ,$$

Bentuk matriks: $|\psi\rangle_{(a)} = U^\dagger |\psi\rangle_{(b)}$.

Transformasi bra:

$$\langle \psi | b_i \rangle = \sum_{j=1}^{n} \langle \psi | a_j \rangle \langle a_j | b_i \rangle = \sum_{j=1}^{n} \langle \psi | a_j \rangle U^{\dagger}_{ji}$$

$$\Longrightarrow \quad {}_{(b)}\langle \psi | = {}_{(a)}\langle \psi | U^{\dagger} \quad ;$$

$$\langle \psi | a_j \rangle = \sum_{k=1}^{n} \langle \psi | b_k \rangle \langle b_k | a_j \rangle = \sum_{k=1}^{n} \langle \psi | b_k \rangle U_{kj}$$

$$\Longrightarrow \quad {}_{(a)}\langle \psi | = {}_{(b)}\langle \psi | U \quad .$$

Perhatian: cara transformasi untuk bra bersongsangan dgn cara transformasi utk ket;
⇒ menjamin sifat ketakvarianan untuk hasildarab terkedalam

$${}_{(b)}\langle \chi | \psi \rangle_{(b)} = {}_{(a)}\langle \chi | U^{\dagger} U | \psi \rangle_{(a)} = {}_{(a)}\langle \chi | \psi \rangle_{(a)} \quad .$$

Teorem: Jika diberi dua set ket asas $\{ | a_i \rangle \}$, $\{ | b_j \rangle \}$, kedua-duanya mematuhi keortonormalan, maka akan wujud satu operator unitari U yang saling menukargantikan set ket asas tersebut tetapi mengekalkan hasildarab terkedalam dalam ruang ket.

Transformasi operator?

Lihat unsurnya

$$O^{(b)}_{ij} = \langle b_i | \hat{O} | b_j \rangle$$

$$= \sum_{k=1}^{n} \sum_{m=1}^{n} \langle b_i | a_k \rangle \langle a_k | \hat{O} | a_m \rangle \langle a_m | b_j \rangle$$

$$= \sum_{k=1}^{n} \sum_{m=1}^{n} U_{ik} O^{(a)}_{km} U^{\dagger}_{mj} \quad .$$

Bentuk matriks:

$$\hat{O}^{(b)} \quad = \quad U \hat{O}^{(a)} U^{\dagger} \qquad .$$

Untuk adjoin:

$$(\hat{O}^{(b)})^{\dagger} \quad = \quad (U \hat{O}^{(a)} U^{\dagger})^{\dagger}$$

$$= \quad U^{\dagger\dagger} (\hat{O}^{(a)})^{\dagger} U^{\dagger} \quad = \quad U (\hat{O}^{(a)})^{\dagger} U^{\dagger} \qquad .$$

$\Longrightarrow$ cara transformasi sama untuk semua operator.

Ketakvarianan hasildarab terkedalam yang melibatkan operator $\hat{O}$:

$$_{(b)}\langle \chi \mid \hat{O}^{(b)} \mid \psi \rangle_{(b)} \quad = \quad _{(a)}\langle \chi \mid U^{\dagger} U \hat{O}^{(a)} U^{\dagger} U \mid \psi \rangle_{(a)} \quad = \quad _{(a)}\langle \chi \mid \hat{O}^{(a)} \mid \psi \rangle_{(a)} \quad .$$

> *Teorem*: Pertimbang dua set ket asas $\{ \mid b_i \rangle \}$, $\{ \mid a_j \rangle \}$ yang terhubung melalui operator unitari U. Mengetahui U, kita boleh bina *jelmaan unitari* bagi satu-satu operator $\hat{O}$ sebagai $U \hat{O} U^{-1}$. Kedua-dua operator $\hat{O}$ dan $U \hat{O} U^{-1}$ dikatakan membentuk *operator-operator setara unitari*.

Pemerhatian kesetaraan operator melalui nilai eigennya: jika $\hat{A} \mid a_i \rangle \; = \; a_i \mid a_i \rangle$,

$$U \hat{A} U^{\dagger} U \mid a_i \rangle \quad = \quad a_i U \mid a_i \rangle$$

$$\Rightarrow \qquad (U \hat{A} U^{\dagger}) \mid b_i \rangle \quad = \quad a_i \mid b_i \rangle \qquad .$$

Surih operator $\quad \mathrm{Tr}(\hat{O}) \; = \; \sum_{i=1}^{n} O_{ii} \; = \; \sum_{i=1}^{n} \langle a_i \mid \hat{O} \mid a_i \rangle \quad$ bersifat tak varian:

$$\sum_{i=1}^{n} \langle a_i \mid \hat{O} \mid a_i \rangle \quad = \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \langle a_i \mid b_j \rangle \langle b_j \mid \hat{O} \mid b_k \rangle \langle b_k \mid a_i \rangle$$

$$= \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \langle b_j \mid \hat{O} \mid b_k \rangle \langle b_k \mid a_i \rangle \langle a_i \mid b_j \rangle \quad = \quad \sum_{j=1}^{n} \langle b_j \mid \hat{O} \mid b_j \rangle \qquad .$$